Mapping Between Antisymmetric Tensor And Weinberg Formulations

By

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Abstract. In the framework of the classical field theory a mapping between antisymmetric tensor matter fields and Weinberg's $2(2j+1)$ component "bispinor" fields is considered. It is shown that such a mapping exists and equations which describe the $j=1$ antisymmetric tensor field coincide with the Hammer-Tucker equations completely and with the Weinberg ones within a subsidiary condition, the Klein-Gordon equation. A new Lagrangian for the Weinberg theory is proposed. It is scalar, Hermitian and contains only the first-order time derivatives of the fields. The remarkable feature of this Lagrangian is the presence of dual field functions, considered as parts of a parity doublet. I study then origins of appearance of the dual solutions in the Weinberg equations on the basis of spinorial analysis and point out the topics which have to be explained in the framework of a secondary quantization scheme.

In the present paper the connection between the Weinberg's $2(2j+1)$- component formulation [1, 2, 3] and the antisymmetric tensor matter field description [4, 5, 6, 7] is studied. The reason is the elaboration of the Bargmann-Wightman-Wigner-type quantum field theory, undertaken by Ahluwalia *et al.* [8] (see also [9]), which brought some hopes on recreation and further development of the Weinberg's $2(2j+1)$ theory. Finding the likely relations between various formulations for $j=1$ (and higher spin) particles could provide a necessary

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basis for both practical phenomenological calculations [10, 11] and experiment.

I start from the Proca equations for a $j=1$ massive particle

$$\partial_\mu F_{\mu\nu} = m^2 A_\nu \quad , \tag{1}$$
$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{2}$$

in the form given by [2, 12]. The Euclidean metric, $x_\mu = (\vec{x}, x_4 = it)$ and notation $\partial_\mu = (\vec{\nabla}, -i\partial/\partial t)$, $\partial_\mu^2 = \vec{\nabla}^{\,2} - \partial_t^2$, are used. By means of the choice of $F_{\mu\nu}$ components as the "physical" variables I re-write the set of equations as

$$m^2 F_{\mu\nu} = \partial_\mu \partial_\alpha F_{\alpha\nu} - \partial_\nu \partial_\alpha F_{\alpha\mu} \tag{3}$$

and

$$\partial_\lambda^2 F_{\mu\nu} = m^2 F_{\mu\nu} \quad . \tag{4}$$

It is easy to show that they can be represented in the form ($F_{44} = 0$, $F_{4i} = iE_i$ and $F_{jk} = \epsilon_{jki} B_i$; $p_\alpha = -i\partial_\alpha$):

$$\begin{cases} (m^2 + p_4^2) E_i + p_i p_j E_j + i\epsilon_{ijk} p_4 p_j B_k = 0 \\ \\ (m^2 + \vec{p}^{\,2}) B_i - p_i p_j B_j + i\epsilon_{ijk} p_4 p_j E_k = 0 \quad , \end{cases} \tag{5}$$

or

$$\begin{cases} \left[m^2 + p_4^2 + \vec{p}^{\,2} - (\vec{J}\vec{p})^2 \right]_{ij} E_j + p_4 (\vec{J}\vec{p})_{ij} B_j = 0 \\ \left[m^2 + (\vec{J}\vec{p})^2 \right]_{ij} B_j + p_4 (\vec{J}\vec{p})_{ij} E_j = 0 \quad . \end{cases} \tag{6}$$

Adding and subtracting the obtained equations yield

$$\begin{cases} m^2 (\vec{E} + i\vec{B})_i + p_\alpha p_\alpha \vec{E}_i - (\vec{J}\vec{p})^2_{ij} (\vec{E} - i\vec{B})_j + p_4 (\vec{J}\vec{p})_{ij} (\vec{B} + i\vec{E})_j = 0 \\ \\ m^2 (\vec{E} - i\vec{B})_i + p_\alpha p_\alpha \vec{E}_i - (\vec{J}\vec{p})^2_{ij} (\vec{E} + i\vec{B})_j + p_4 (\vec{J}\vec{p})_{ij} (\vec{B} - i\vec{E})_j = 0 \quad , \end{cases} \tag{7}$$

where $(\vec{J}_i)_{jk} = -i\epsilon_{ijk}$ are the $j=1$ spin matrices. Equations are equivalent (within a factor 1/2) to the Hammer-Tucker equation [3], see also [11, 13]:

$$(\gamma_{\alpha\beta} p_\alpha p_\beta + p_\alpha p_\alpha + 2m^2) \psi_1 = 0 \quad , \tag{8}$$

in the case of a choice $\chi = \vec{E} + i\vec{B}$ and $\varphi = \vec{E} - i\vec{B}$, $\psi_1 = \mathrm{column}(\chi, \quad \varphi)$. Matrices $\gamma_{\alpha\beta}$ are the covariantly defined matrices of Barut, Muzinich and Williams [14]. One can check that the equation (8) is characterized by solutions with a physical dispersion only, but some points concerned with massless limit should be clarified properly.

Following to the analysis of ref. [9b, p.1972], one can conclude that other equations with the correct physical dispersion could be obtained from

$$(\gamma_{\alpha\beta} p_\alpha p_\beta + a p_\alpha p_\alpha + b m^2) \psi = 0 \quad . \tag{9}$$

As a result of taking into account $E^2 - \vec{p}^{\,2} = m^2$ we draw the conclusion that there exists an infinity number of appropriate equations provided that b and a are connected as follows:

$$\frac{b}{a+1} = 1 \qquad \text{or} \qquad \frac{b}{a-1} = 1 \quad .$$

However, there are only two equations that do not have acausal tachyonic solutions. The second one (with $a = -1$ and $b = -2$) is

$$(\gamma_{\alpha\beta} p_\alpha p_\beta - p_\alpha p_\alpha - 2m^2)\psi_2 = 0 \quad . \tag{10}$$

Thus, we found the "double" of the Hammer-Tucker equation. In the tensor form it leads to the equations dual to (5):

$$\begin{cases} (m^2 + \vec{p}^{\,2}) C_i - p_i p_j C_j - i\epsilon_{ijk} p_4 p_j D_k = 0 \\ \\ (m^2 + p_4^2) D_i + p_i p_j D_j - i\epsilon_{ijk} p_4 p_j C_k = 0 \end{cases} \quad , \tag{11}$$

which could be re-written in the form, *cf.* (3),

$$m^2 \tilde{F}_{\mu\nu} = \partial_\mu \partial_\alpha \tilde{F}_{\alpha\nu} - \partial_\nu \partial_\alpha \tilde{F}_{\alpha\mu} \quad , \tag{12}$$

with $\tilde{F}_{4i} = iD_i$ and $\tilde{F}_{jk} = -\epsilon_{jki} C_i$. C_i is an analog of E_i and D_i is an analog of B_i; $\tilde{F}_{\mu\nu} = \frac{1}{2}\epsilon_{\mu\nu\rho\sigma} F_{\rho\sigma}$, $\epsilon_{1234} = -i$. I have used above the following properties of the antisymmetric Levi-Civita tensor

$$\epsilon_{ijk}\epsilon_{ijl} = 2\delta_{kl} \quad , \quad \epsilon_{ijk}\epsilon_{ilm} = (\delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}) \quad ,$$

and

$$\epsilon_{ijk}\epsilon_{lmn} = \mathrm{Det}\begin{pmatrix} \delta_{il} & \delta_{im} & \delta_{in} \\ \delta_{jl} & \delta_{jm} & \delta_{jn} \\ \delta_{kl} & \delta_{km} & \delta_{kn} \end{pmatrix} \quad .$$

Comparing the structure of the Weinberg equation ($a = 0$, $b = 1$) with the Hammer-Tucker "doubles" one can convince ourselves that the former can be represented in the tensor form:

$$m^2 F_{\mu\nu} = \partial_\mu \partial_\alpha F_{\alpha\nu} - \partial_\nu \partial_\alpha F_{\alpha\mu} + \frac{1}{2}(m^2 - \partial_\lambda^2) F_{\mu\nu} \quad . \tag{13}$$

However, as we learnt, it is possible to build a "double" equation:

$$m^2 \tilde{F}_{\mu\nu} = \partial_\mu \partial_\alpha \tilde{F}_{\alpha\nu} - \partial_\nu \partial_\alpha \tilde{F}_{\alpha\mu} + \frac{1}{2}(m^2 - \partial_\lambda^2) \tilde{F}_{\mu\nu} \quad . \tag{14}$$

Thus, the Weinberg's set of equations could be written in the form:

$$(\gamma_{\alpha\beta} p_\alpha p_\beta + m^2)\psi_1 = 0 \quad , \tag{15}$$

$$(\gamma_{\alpha\beta} p_\alpha p_\beta - m^2)\psi_2 = 0 \quad . \tag{16}$$

Thanks to the Klein-Gordon equation (4) these equations are equivalent to the Proca tensor equations (and to the Hammer-Tucker ones) in a free case. However, if interaction is included,

one cannot say that. The general solution describing a $j = 1$ particle is presented as a superposition

$$\Psi^{(1)} = c_1 \psi_1^{(1)} + c_2 \psi_2^{(1)} \quad , \tag{17}$$

where the constants c_1 and c_2 are to be defined from the boundary, initial and normalization conditions.

Let me note a surprising fact: while both the Proca equations (or the Hammer-Tucker ones) and the Klein-Gordon equation do not possess "non-physical" solutions, their sum, Eqs. (13,14) or the Weinberg equations (15,16), acquires tachyonic solutions. For the following it is also useful to note some remarkable features of this set of equations. Equations (15) and (16) could be re-casted in another form (index "T" denotes a transpose matrix):

$$\left[\gamma_{44} p_4^2 + 2\gamma_{4i}^T p_4 p_i + \gamma_{ij} p_i p_j - m^2\right] \psi_1^{(2)} = 0 \quad , \tag{18}$$

$$\left[\gamma_{44} p_4^2 + 2\gamma_{4i}^T p_4 p_i + \gamma_{ij} p_i p_j + m^2\right] \psi_2^{(2)} = 0 \quad , \tag{19}$$

respectively, if understand $\psi_1^{(2)} = \text{column}(B_i + iE_i, \quad B_i - iE_i) = i\gamma_5\gamma_{44}\psi_1^{(1)}$ and $\psi_2^{(2)} = \text{column}(D_i + iC_i, \quad D_i - iC_i) = i\gamma_5\gamma_{44}\psi_2^{(1)}$. The general solution is again a linear combination

$$\Psi^{(2)} = c_1 \psi_1^{(2)} + c_2 \psi_2^{(2)} \quad . \tag{20}$$

From, *e.g.*, Eq. (15), dividing $\psi_1^{(1)}$ into longitudinal and transversal parts one can come to the equations:

$$\begin{aligned} &\left[\mathcal{E}^2 - \vec{p}^{\,2}\right] (\vec{E} + i\vec{B})^{\parallel} - m^2 (\vec{E} - i\vec{B})^{\parallel} + \\ &+ \quad \left[\mathcal{E}^2 + \vec{p}^{\,2} - 2\mathcal{E}(\vec{J}\vec{p})\right] (\vec{E} + i\vec{B})^{\perp} - m^2 (\vec{E} - i\vec{B})^{\perp} = 0 \quad , \end{aligned} \tag{21}$$

$$\begin{aligned} &\left[\mathcal{E}^2 - \vec{p}^{\,2}\right] (\vec{E} - i\vec{B})^{\parallel} - m^2 (\vec{E} + i\vec{B})^{\parallel} + \\ &+ \quad \left[\mathcal{E}^2 + \vec{p}^{\,2} + 2\mathcal{E}(\vec{J}\vec{p})\right] (\vec{E} - i\vec{B})^{\perp} - m^2 (\vec{E} + i\vec{B})^{\perp} = 0 \quad . \end{aligned} \tag{22}$$

Therefore, in classical field theory antisymmetric tensor matter fields are the fields with the transversal components in massless limit (*cf.* with a quantized case, ref. [5, 6, 7, 11] and with the remark in the end of the paper).[2]

Under the transformations $\psi_1^{(1)} \rightarrow \gamma_5 \psi_2^{(1)}$ or $\psi_1^{(2)} \rightarrow \gamma_5 \psi_2^{(2)}$ the set of equations (15) and (16), or (18) and (19), are invariant. The origin of this fact is the dual invariance of the Proca equations. In the matrix form dual transformations correspond to the chiral transformations (see about these relations, *e.g.*, ref. [15]).

Another equation has been proposed in refs. [2, 8]

$$(\gamma_{\alpha\beta} p_\alpha p_\beta + \wp_{u,v} m^2)\psi = 0 \quad , \tag{23}$$

[2]Let me also mention that the equations (4.21) and (4.22) of the paper [1b,p.B888] may be insufficient for describing a $j = 1$ massless field as noted absolutely correctly in refs. [9b], see also [11c,e]. As a matter of fact their inadequate application leads to speculations on the violation of the Correspondence Principle. Unfortunately, the author of ref. [1b] missed the fact that the matrix $(\vec{J}\vec{p})$ has no the inverse one.

where $\wp_{u,v} = i(\partial/\partial t)/E$, what distinguishes u- (positive-energy) and v- (negative-energy) solutions. For instance, in [8a,footnote 4] it is claimed that

$$\psi_\sigma^+(x) = \frac{1}{(2\pi)^3} \int \frac{d^3p}{2\omega_p} u_\sigma(\vec{p}) e^{ipx} \quad , \tag{24}$$

$\omega_p = \sqrt{m^2 + \vec{p}^2}$, $p_\mu x_\mu = \vec{p}\vec{x} - Et$, must be described by the equation (15), in the meantime,

$$\psi_\sigma^-(x) = \frac{1}{(2\pi)^3} \int \frac{d^3p}{2\omega_p} v_\sigma(\vec{p}) e^{-ipx} \quad , \tag{25}$$

from equation (16). The analysis of this question and the comparison of the models based on the set of equations obtained here and in ref. [8] (as well as the discussion of consequences of new constructs in the $(j,0)\oplus(0,j)$ representation space and their differences from the papers of [1, 2, 3]) are left for further publications.

Let me consider the question of the "double" solutions on the ground of spinorial analysis. In ref. [2, p.1305] (see also [16, p.60-61]) relations between the Weinberg bispinor (bivector, indeed) and symmetrical spinors of $2j$ rank have been discussed. It was noted there: "*The wave function may be written in terms of two three-component functions $\psi = column(\chi \quad \varphi)$, that, for the continuous group, transform independently of each other and that are related to two symmetrical spinors:*

$$\chi_1 = \chi_{\dot{1}\dot{1}}, \quad \chi_2 = \sqrt{2}\chi_{\dot{1}\dot{2}}, \quad \chi_3 = \chi_{\dot{2}\dot{2}} \quad , \tag{26}$$

$$\varphi_1 = \varphi^{11}, \quad \varphi_2 = \sqrt{2}\varphi^{12}, \quad \varphi_3 = \varphi^{22} \quad , \tag{27}$$

when the standard representation for the spin-one matrices, with S_3 diagonal is used."

Under the inversion operation we have the following rules [16, p.59]: $\varphi^\alpha \to \chi_{\dot{\alpha}}$, $\chi_{\dot{\alpha}} \to \varphi^\alpha$, $\varphi_\alpha \to -\chi^{\dot{\alpha}}$ and $\chi^{\dot{\alpha}} \to -\varphi_\alpha$. Hence, we deduce (if understand $\chi_{\dot{\alpha}\dot{\beta}} = \chi_{\{\dot{\alpha}}\chi_{\dot{\beta}\}}$, $\varphi^{\alpha\beta} = \varphi^{\{\alpha}\varphi^{\beta\}}$)

$$\chi_{\dot{1}\dot{1}} \to \varphi^{11} \quad , \quad \chi_{\dot{2}\dot{2}} \to \varphi^{22} \quad , \quad \chi_{\{\dot{1}\dot{2}\}} \to \varphi^{\{12\}} \quad , \tag{28}$$

$$\varphi^{11} \to \chi_{\dot{1}\dot{1}} \quad , \quad \varphi^{22} \to \chi_{\dot{2}\dot{2}} \quad , \quad \varphi^{\{12\}} \to \chi_{\{\dot{1}\dot{2}\}} \quad . \tag{29}$$

However, this definition of symmetrical spinors of the second rank χ and φ is ambiguous. We are also able to define $\tilde{\chi}_{\dot{\alpha}\dot{\beta}} = \chi_{\{\dot{\alpha}}H_{\dot{\beta}\}}$ and $\tilde{\varphi}^{\alpha\beta} = \varphi^{\{\alpha}\Phi^{\beta\}}$, where $H_{\dot{\beta}} = \varphi^*_\beta$, $\Phi^\beta = (\chi^{\dot{\beta}})^*$. It is easy to show that in the framework of the second definition we have under the inversion operation:

$$\tilde{\chi}_{\dot{1}\dot{1}} \to -\tilde{\varphi}^{11} \quad , \quad \tilde{\chi}_{\dot{2}\dot{2}} \to -\tilde{\varphi}^{22} \quad , \quad \tilde{\chi}_{\{\dot{1}\dot{2}\}} \to -\tilde{\varphi}^{\{12\}} \quad , \tag{30}$$

$$\varphi^{11} \to -\tilde{\chi}_{\dot{1}\dot{1}} \quad , \quad \tilde{\varphi}^{22} \to -\tilde{\chi}_{\dot{2}\dot{2}} \quad , \quad \tilde{\varphi}^{\{12\}} \to -\tilde{\chi}_{\{\dot{1}\dot{2}\}} \quad . \tag{31}$$

The Weinberg bispinor $(\chi_{\dot{\alpha}\dot{\beta}} \quad \varphi^{\alpha\beta})$ corresponds to the equations (18) and (19) , meanwhile $(\tilde{\chi}_{\dot{\alpha}\dot{\beta}} \quad \tilde{\varphi}^{\alpha\beta})$, to the equation (15) and (16).

Similar conclusions can be drawn in the case of the parity definition as $P^2 = -1$. Transformation rules are then $\varphi^\alpha \to i\chi_{\dot{\alpha}}$, $\chi_{\dot{\alpha}} \to i\varphi^\alpha$, $\varphi_\alpha \to -i\chi^{\dot{\alpha}}$ and $\chi^{\dot{\alpha}} \to -i\varphi_\alpha$, ref. [16, p.59] . Hence, $\chi_{\dot{\alpha}\dot{\beta}} \leftrightarrow -\varphi^{\alpha\beta}$ and $\tilde{\chi}_{\dot{\alpha}\dot{\beta}} \leftrightarrow -\tilde{\varphi}^{\alpha\beta}$, but $\varphi^\alpha{}_\beta \leftrightarrow \chi_{\dot{\alpha}}{}^{\dot{\beta}}$ and $\tilde{\varphi}^\alpha{}_\beta \leftrightarrow \tilde{\chi}_{\dot{\alpha}}{}^{\dot{\beta}}$.

Next, one can propose, *e.g.*, the following Lagrangian for physical fields $\psi_1^{(1)}$ and $\psi_2^{(2)}$ chosen:

$$\mathcal{L} = \partial_\mu \psi_1^\dagger \widetilde{\gamma}_{\mu\nu} \partial_\nu \psi_2 + \partial_\mu \psi_2^\dagger \gamma_{\mu\nu} \partial_\nu \psi_1 + m^2 \psi_2^\dagger \psi_1 + m^2 \psi_1^\dagger \psi_2 \quad . \tag{32}$$

It is scalar (as opposed to ref. [11c]), Hermitian (*cf.* ref. [17]) and of the first order in time derivatives of fields (as opposed to ref. [8b]). It is easy to check that $\widetilde{\gamma}_{\mu\nu} = \gamma_{44}\gamma_{\mu\nu}\gamma_{44}$ transforms as a tensor, therefore the Lagrangian is still scalar. It leads to the equations which have solutions in spite of the fact that a procedure of adding the Hermitian-conjugated part in previous attempts led to an inconsistent theory (when the Euclidean metric was used), ref. [17]. I would still like to notice that two dual field functions are used and they are considered as independent ones in the present formulation. The Lagrangian (32) leads to the equations of another set of Weinberg "doubles", each of those has solution.[3] In the case of the use of a pseudoeuclidean metric (when γ_{0i} is chosen to be anti-Hermitian) it is possible to write the Lagrangian following for F. D. Santos and H. Van Dam, ref. [18b]:

$$\mathcal{L} = \partial^\mu \bar{\psi} \gamma_{\mu\nu} \partial^\nu \psi - m^2 \bar{\psi}\psi \quad . \tag{33}$$

It does not lead to the difficulties noted by W. Greiner and H. M. Ruck (since there is no necessity to add the Hermitian-conjugated part). However, in the papers of F. Santos and H. Van Dam, ref. [18], the possibility of appearance of the "doubles" has not been considered (neither in any other paper on the $2(2j+1)$ formalism, to my knowledge).

I conclude: Both the theory of Ahluwalia *et al.* and the model based on the use of ψ_1 and ψ_2 are connected with the antisymmetric tensor matter field description. The models are both possible at the classical level. However, even at the classical level they may lead to different predictions, which also differ from the previous considerations. They have to be quantized consistently. Special attention should be paid to the translational and rotational invariance (the conservation of energy-momentum and angular momentum, indeed), the interaction representation, causality, locality and covariance of theory, *i.e.* to all topics, which are the axioms of the modern quantum field theory [19, 20]. A consistent theory has also to take into account the degeneracy of states.[4]

Then, I would like to draw reader's attention at the problem put forward in [11c]. The papers [5, 6, 7] have proved that the quantized $F_{\mu\nu}$ tensor field describes the particles with the longitudinal component only in the massless limit, the 0^+ particle. In the meantime, the quantized $\tilde{F}_{\mu\nu}$ field describes the 0^- massless particle.[5] How is the Weinberg theorem[6] to be treated in this case (we use the $(1,0)+(0,1)$ representation, however, $\lambda = 0$)? Why do the Weinberg equations seem to describe the transversal fields in the classical theory and the longitudinal fields in the quantized theory. Finally, since this formulation appears

[3]On first sight the second equation obtained from this Lagrangian differs from the Weinberg "double", Eq. (16). However, let us not forget the discussion surrounding Eq. (19).

[4]Namely, two dual functions ψ_1 and ψ_2 (or $F_{\mu\nu}$ and $\tilde{F}_{\mu\nu}$, the "doubles") are considered to yield the same spectrum. Unfortunately, in previous formulations of the dual theories many specific features of such a consideration have not been taken into account. I am going to present a detailed version of this Weinberg-type model in the forthcoming publications.

[5]Not all the authors agreed with this conclusion, *e.g.*, [4, 22, 8].

[6]Let me recall that the Weinberg theorem states: *The fields constructed from massless particle operator* $a(\vec{p}, \lambda)$ *of the definite helicity transform according to representation* (A, B) *such that* $B - A = \lambda$.

to be related[7] to the dual theories let me reproduce some references which could be useful in further development of the theory. Dual formulations of electrodynamics and massive vector theory based on $F_{\mu\nu}$ and $\tilde{F}_{\mu\nu}$ have been considered in refs. [15, 21, 22, 23, 24]. The interaction of the Dirac field with the dual fields $F_{\mu\nu}$ and $\tilde{F}_{\mu\nu}$ has been considered in ref. [25] (what, indeed, implies existence of the anomalous electric dipole moment of a fermion). The dual formulation of the Dirac field has also been considered, *e. g.*, ref. [26] (see also [27]).

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Recently after writing this paper I learnt about the papers of Prof. M. W. Evans published in FPL, FP and Physica B as well as about his several books. As a matter of fact they deal (from a different viewpoint) with the questions put forward in my series. I am grateful to Prof. Evans for many private communications on his concept of the longitudinal $B^{(3)}$ field of electromagnetism and I appreciate his encouragements.

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